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Some properties for a subclass of Bazilevič functions*

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ABSTRACT Let S represents the class of functions f which are analytic and univalent in the unit disc $D=\{z:|z|<1\}$ with $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. We denote B to be the class of Bazilevič functions which to date forms the largest subclass of S. In this paper we introduce the class $B_1^{\lambda}(\alpha,\beta)$, a subclass of B and give some interesting properties for this class, in particular results concerning iterated integral operators.

ABSTRAK Katakan S melambangkan kelas fungsi f yang analisis dan univalen pada cakera unit $D=\{z:|z|<1\}$ dengan $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Diketahui bahawa B yang mewakili kelas fungsi Bazilevič merupakan subkelas terbesar S. Kelas $B_1^{\lambda}(\alpha,\beta)$ diperkenalkan serta dibincangkan beberapa ciri menarik mengenai kelas ini, khususnya operator pengamir yang terlelar.

(univalent functions, Bazilevič functions, iterated integral operators)

INTRODUCTION

Let A denote the class of functions f analytic in the unit disc $D=\{z:|z|<1\}$ with $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and S, a subclass of A which consists of functions univalent in D.

A function $f\in A$ is said to be starlike of order β if and only if for $z\in D$

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta$$

for some β $(0\le\beta<1)$. We denote such a class by $S^*(\beta)$. This class was introduced by Robertson [9] and studied further by others including Schild [10] and Jack [5].

A function $f\in A$ is said to be Bazilevič of type α, δ if α and δ are any real numbers with $\alpha>0$ and f has the following representation

$$f(z)=\left((\alpha+i\delta)\int_0^z p(t)g(t)^{\alpha}t^{(i\delta-1)}dt\right)^{\frac{1}{\alpha+i\delta}} \qquad (1)$$

for any $z\in D$, $g\in S^*(0)$ and p satisfies the conditions $p(0)=1$ and $\operatorname{Re}p(z)>0$ $(z\in D)$. (Powers in (1) are principal values). In 1955, Bazilevič [1] showed that the class B consisting of Bazilevič functions forms a subclass of S. Amongst others, Sheil-Small [11] and Pommerenke [8] looked into this class B. For $\delta=0$, the subclass $B(\alpha)$ of B is obtained and putting $g(z)\equiv z$, we have the subclass $B_1(\alpha)$. Zamorski [15], Thomas [13] and Singh [12] looked into $B(\alpha)$ to obtain interesting properties on it. In the same paper, Singh [loc. cit.] also gives results on the class $B_1(\alpha)$. Other authors which also considered these and other subclasses of B includes Obradovic and Owa [6], [7], Thomas [14] and Halim [2].

From (1), it easily follows that $f\in B(\alpha)$ if, and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)^{1-\alpha}g(z)^{\alpha}}\right)>0 \qquad (z\in D),$$

and thus $f\in B_1(\alpha)$, if, and only if

$$\operatorname{Re}\left(\frac{z^{1-\alpha}f'(z)}{f(z)^{1-\alpha}}\right)>0 \qquad (z\in D).$$

We now extend the class $B_1(\alpha)$ to $B_1(\alpha,\beta)$ as follows:

Definition 1.1: Let $f \in A \cdot f \in B_1(\alpha,\beta)$ if, and only if

$$\operatorname{Re}\left(\frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}}\right) > \beta \qquad (z \in D)$$

for $\alpha > 0$ and $0 \le \beta < 1$.

Further, we introduce a new subclass of B via the following definition:

Definition 1.2: Let $f \in A$. Then $f \in B_1^{\lambda}(\alpha,\beta)$ if, and only if

$$[f(z)]^{\alpha} = (1-\lambda) z^{\alpha} + \lambda [h(z)]^{\alpha} \qquad (z \in D)$$

for some fixed constant λ with $0 < \lambda \le 1$ and h belongs to $B_1(\alpha,\beta)$.

Remark. $f \in B_1^{\lambda}(\alpha,\beta)$ is normalized and $B_1(\alpha,\beta) \equiv B_1^{1}(\alpha,\beta)$.

2. Iterated Integral Operator

The best estimate for the lower bound of $\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\alpha}$ was established for $f \in B_1(\alpha)$ by the author and Thomas [4]. This result was further used in obtaining estimates for the real part of some iterated integral operators. In using the same approach we now extend these results to $B_1^{\lambda}(\alpha,\beta)$.

Definition 2.1: For $z \in D$, $a > -1$ and $n = 1,2,3,\ldots$, define the iterated integral operator I_n as

$$I_n(z) = \frac{a+1}{z^{a+1}} \int_0^z t^a I_{n-1}(t)\,dt \qquad (2)$$

where $I_0(z) = (f(z)/z)^{\alpha}$.

Theorem 2.1. Let $f \in B_1^{\lambda}(\alpha,\beta)$ and $z = re^{i\theta} \in D$. Then for $n \ge 0$,

$$\operatorname{Re} I_n(z) \ge \sigma_n(r) > \sigma_n(1) \qquad (3)$$

and $\sigma_n(r) < 1$

where

$$\sigma_n(r) = \frac{a+1}{r^{a+1}} \int_0^r \rho^a \sigma_{n-1}(\rho)\,d\rho \qquad \text{for } n = 1,2,3,\ldots$$

and

$$\sigma_0(r) = (1-\lambda)(1-\beta) + \frac{\lambda\alpha(1-\beta)}{r^{\alpha}} \int_0^r \rho^{\alpha-1}\left(\frac{1-\rho}{1+\rho}\right) d\rho + \beta \cdot$$

Equality is attained for f_0 defined by

$$f_0^{\alpha}(z) = \alpha \int_0^z t^{\alpha-1}\left(\frac{(1-t)(1-\beta)}{(1+t)} + (1-\lambda+\beta)\right) dt \cdot$$

To prove the above Theorem, we require the following Lemma.

Lemma 2.1[3]. Let Γ_n be defined as given below

$$\Gamma_n(r) = \frac{a+1}{r^{a+1}} \int_0^r \rho^a \Gamma_{n-1}(\rho)\,d\rho \qquad \text{for } n = 1,2,3,\ldots$$

and

$$\Gamma_0(r) = \frac{\alpha(1-\beta)}{r^{\alpha}} \int_0^r \rho^{\alpha-1}\left(\frac{1-\rho}{1+\rho}\right) d\rho \cdot$$

Then for fixed r $(0 < r < 1)$ and $n \ge 1$, $\Gamma_{n-1}(r) < \Gamma_n(r)$.

Proof (Theorem 2.1). We use induction to prove the theorem. First, consider the case $n = 0$. For $f \in B_1^{\lambda}(\alpha,\beta)$, $\exists\, h \in B_1(\alpha,\beta)$ such that

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\alpha} = 1 - \lambda + \lambda \operatorname{Re}\left(\frac{h(z)}{z}\right)^{\alpha}.$$

Thus, using the results in [3], i.e. for $h \in B_1(\alpha,\beta)$ with $z = re^{i\theta}$

$$\operatorname{Re}\left(\frac{h(z)}{z}\right)^{\alpha} \ge \frac{\alpha}{r^{\alpha}} \int_0^r \rho^{\alpha-1}\left(\frac{(1-\beta)(1-\rho)}{1+\rho} + \beta\right) d\rho,$$

we obtain

$$\operatorname{Re} I_0(z) \geq 1-\lambda+\frac{\lambda\alpha}{r^\alpha}\int_0^r \rho^{\alpha-1}\left[\frac{(1-\beta)(1-\rho)}{1+\rho}+\beta\right]d\rho$$
$$=1+2\lambda\alpha(1-\beta)\sum_{k=1}^{\infty}\frac{(-r)^k}{k+\alpha}$$
$$=\sigma_0(r).$$

Elementary calculations will show that $\sigma_0(1)<\sigma_0(r)<1$ for $0<\rho<r<1$. Next, from (2) with $t=\rho e^{i\theta}$,

$$\operatorname{Re} I_{n+1}(z)=\operatorname{Re}\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a I_n(t)dt\right\}$$
$$=\frac{a+1}{r^{a+1}}\int_0^r \rho^a \operatorname{Re} I_n(\rho e^{i\theta})d\rho$$
$$\geq\frac{a+1}{r^{a+1}}\int_0^r \rho^a\sigma_n(\rho)d\rho$$
$$=\sigma_{n+1}(r),$$

where by induction the first inequality in (3) is established.

For $n\geq 1$, the series

$$\sigma_n(r)=1+2\lambda\alpha(1-\beta)(1+a)^n\sum_{j=1}^{\infty}\frac{(-r)^j}{(j+\alpha)(j+a+1)^n}$$

is absolutely convergent and writing it is as follows

$$\sigma_n(r)=1-2\lambda\alpha(1-\beta)(1+a)^n\left(\frac{r}{(1+\alpha)(2+a)^n}-\frac{r^2}{(2+\alpha)(3+a)^n}\right)$$
$$-2\lambda\alpha(1-\beta)(1+a)^n\left(\frac{r^3}{(3+\alpha)(4+a)^n}-\frac{r^4}{(4+\alpha)(5+a)^n}\right)+$$
$$=1-2\lambda\alpha(1-\beta)(1+a)^n X(r,\alpha)$$

where

$$X(r,\alpha)=\sum_{k=1}^{\infty}\left(\frac{r^{2k-1}}{(2k-1+\alpha)(2k+a)^n}-\frac{r^{2k}}{(2k+\alpha)(2k+1+a)^n}\right)$$

shows that $\sigma_n(r)<1$.

Finally, we show $\sigma_n(r)>\sigma_n(1)$. Noting that $\sigma_n(r)=(1-\beta)(1-\lambda)+\beta+\lambda\Gamma_n(r)$ for $n=0,1,2,3,\ldots$ and using Lemma 2.1, we then have for a fixed r, $\sigma_{n-1}(r)<\sigma_n(r)$. Also, since

$$r\sigma_n'(r)=(1+a)[\sigma_{n-1}(r)-\sigma_n(r)],$$

this shows that for a fixed $n\geq 1$, $r\sigma_n'(r)<0$ indicating a decreasing function $\sigma_n(r)$. This completes the proof of the Theorem 2.1.

3. Other Properties

In this section we list other properties of the class $B_1^\lambda(\alpha,\beta)$.

Theorem 3.1. $B_1^\lambda(\alpha,\beta)\subseteq B_1(\alpha,\beta)$.

Proof. For any $f\in B_1^\lambda(\alpha,\beta)$, $\exists\, h\in B_1(\alpha,\beta)$ such that $[f(z)]^\alpha=(1-\lambda)z^\alpha+\lambda[h(z)]^\alpha$.

Differentiating w.r.t. z and further multiplying with $z^{1-\alpha}$ gives

$$z^{1-\alpha}\alpha[f(z)]^{\alpha-1}f'(z)=z^{1-\alpha}\alpha(1-\lambda)z^{\alpha-1}$$
$$+z^{1-\alpha}\lambda\alpha[h(z)]^{\alpha-1}h'(z)$$

$$\Rightarrow\quad \operatorname{Re}\left(\frac{z^{1-\alpha}f'(z)}{f(z)^{1-\alpha}}\right)=1-\lambda+\lambda\operatorname{Re}\left(\frac{z^{1-\alpha}h'(z)}{h(z)^{1-\alpha}}\right)$$
$$>1-\lambda+\lambda\beta$$
$$>\beta$$
$$\therefore\quad f\in B_1(\alpha,\beta)$$

Theorem 3.2. If $f\in B_1^\lambda(\alpha,\beta)$ then the function g given as $[g(z)]^{\alpha+\mu}=z^\mu[f(z)]^\alpha$ for $\mu\geq 0$ belong to $B_1^\lambda(\alpha+\mu,\beta)$.

Proof. For $\mu\geq 0$, and since $f\in B_1^\lambda(\alpha,\beta)$ we have

$$[g(z)]^{\alpha+\mu}=z^\mu[f(z)]^\alpha$$
$$=z^\mu\{(1-\lambda)z^\alpha+\lambda[h(z)]^\alpha\}$$
$$=(1-\lambda)z^{\alpha+\mu}+\lambda z^\mu h(z)]^\alpha$$

for some $h\in B_1(\alpha,\beta)$. Obradovic and Owa in [7] proved that for $\mu\geq 0$,

$$h\in B_1(\alpha,\beta)\Rightarrow z^\mu[h(z)]^\alpha\in B_1(\alpha+\mu,\beta),$$

thus this means $g\in B_1^\lambda(\alpha+\mu,\beta)$.

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