PHASE EQUIVALENT ENERGY-DEPENDENT POTENTIALS-A SUPERSYMMETRIC APPROACH

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ABSTRACT Phase equivalent potential corresponding to an energy-dependent potential is constructed via the formalism of supersymmetric quantum mechanics and the merit of our approach is assessed through model calculations.

Keywords: Supersymmetry, Phase equivalent potential, Phase function method, scattering phase shift

1. INTRODUCTION

The Hamiltonian hierarchy problems are generally dealt with using the supersymmetric quantum (SUSY) mechanics (Witten 1981; Cooper & Freedman 1983; Richard, Haymaker & Rau 1985; Laha, Bhattacharyya, Roy & Talukdar 1988). The Hamiltonian hierarchy problems lead to the addition of centrifugal barrier to the parent potential and as a result the higher partial wave interactions are reproduced quite accurately. The interest in this area is triggered initially through the connection between degeneracy and symmetry. The methodology of SUSY quantum mechanics, identical to that of supersymmetry in field theory, was first suggested by Witten (1981), which narrates that a compatible Hamiltonian may be constructed for any parent Hamiltonian resulting in a supersymmetric Hamiltonian (Laha, Bhattacharyya, Roy & Talukdar 1988).

The application of supersymmetry (SUSY) in quantum mechanics opens a new direction to the problem of generating phase-equivalent potentials. According to the Levinson theorem (Newton 1982), the well behaved potentials with different numbers of bound states do not provide the same phase shifts at all energies. But the singular potentials with singularity obey the generalized Levinson theorem and reproduce phase equivalent potentials with different numbers of bound states. This duality of the 'deep' and 'shallow' type potentials (Baye 1987) are common in nuclear physics particularly in nucleus-nucleus scattering and optical model calculations. A number of transformations exist in SUSY quantum mechanics which either add or remove a number of bound states to/from the spectrum of a given Hamiltonian keeping the phase shifts unaltered.

The separable nonlocal interactions have been used successfully to fit nucleon-nucleon phase data for different angular momentum states.

Phase equivalent energy-dependent local potentials to nonlocal ones have been constructed by several groups to study nucleon-nucleon and nucleus-nucleus scattering problems (Arnold and MacKellar 1971; Laha & Bhoi 2014; Laha, Majumder & Bhoi 2018). In this model the characteristics of the nonlocal interaction is compared with the phenomenology of the local potential, particularly the development of repulsive core and the phase shifts. The aim of the paper is to provide a compact formula for phase-equivalent potentials and their related wave functions within the formalism of SUSY quantum mechanics. Arnold and MacKellar (1971) developed a method for constructing phase equivalent local potential from a separable nonlocal one. We shall follow the method of Arnold & MacKellar (1971) to construct phase equivalent energy-dependent interactions and the associated local wave function. This local wave function in turn is used to construct phase equivalent potential through SUSY formalism. Several groups (Buck, Friedrich & Wheatley 1977; Michel et al., 1983) studied the supersymmetric aspects of the nucleon-nucleon scattering with one pion exchange model and examined the equivalence of deep parent potential to a shallow one with repulsive core.

In the next section we construct phase equivalent potential to energy-dependent local Yamaguchi potential. The third section is devoted to judge the merit of our approach through some model calculations by judicious use of the phase function method (PFM). The results obtained in this process are equivalent to the earlier results and finally, we conclude the results at the end.

2. PHASE EQUIVALENT POTENTIAL

Using the SUSY transformation which introduces a new ground state to V(r) below its ground state, the resultant phase equivalent potential is obtained as (Sukumar 1985),

$$V_{12}(r) = V(r) - \frac{d^2}{dr^2} \ln\left[1 + \chi \int_0^r \left(\varphi^{(0)}(x)\right)^2 dx\right], \quad (1)$$

where χ is an adjustable parameter and $\varphi^{(0)}(x)$ denotes the near the origin behaviour of the ground state wave function of $V(r)$. The resultant potentials $V_{12}(r)$ in equation (1) for $\infty > \chi > -1$ possess identical spectra; same phase shifts and normalization constants for the excited states. However, they have different normalization constants for the ground state with different values of χ. Hence this group of potentials belongs to a phase equivalent family. These expressions for the new group of potentials and eigen functions are in close agreement with that of the Gelfand–Lavitan (1955) procedure for changing the normalization constant of the ground state (Sukumar 1987; Bargmann 1949). Here in the present work we have chosen to work with $\chi = 1$.

About forty five years ago Arnold and MacKellar (1971) proposed a method for constructing phase equivalent energy-dependent local potential to a non-local one. Solutions of a non-local equation is related to the solutions of the equivalent local potentials by

$$\Phi_{Nonlocal}(k,r) = B(k,r)\Phi_{Local}(k,r) \tag{2}$$

and

$$f_{Nonlocal}(k,r) = B(k,r) f_{Local}(k,r), \tag{3}$$

where $\Phi_{Nonlocal}(k,r)$ and $f_{Nonlocal}(k,r)$ are the regular and irregular solutions of the parent nonlocal Yamaguchi [1954) potential. The function $B(k,r)$ is the damping function and is related to the Wronskian $J(k,r)$ of the pair of irregular solutions $f_{Nonlocal}(\pm k,r)$ which is written as $B(k,r) = J(k,r)^{\frac{1}{2}}$ where,

$$\begin{aligned} J(k,r) = (2ik)^{-1}[\, & f_{Nonlocal}(-k,r).f'_{Nonlocal}(k,r) \\ & - f_{Nonlocal}(k,r) f'_{Nonlocal}(-k,r)] \end{aligned} \tag{4}$$

with $f_{Nonlocal}(-k,r) = f_{Nonlocal}{}^{*}(k,r)$ Following the approach of Arnold and MacKellar (1971) equivalent local potential $V(k,r)$is obtained as

$$\begin{aligned} V(k,r) = & -\frac{1}{2}\frac{J''}{J} + \frac{3}{4}\left(\frac{J'}{J}\right)^2 \\ & -\frac{1}{J}\int_0^\infty V(r,s)\frac{d}{dr}Q(k,r,s)ds \end{aligned} \tag{5}$$

where

$$\begin{aligned} Q(k,r,s) = & \frac{1}{2ik}[\, f_{Nonlocal}(-k,r).f_{Nonlocal} \\ & (k,s) - f_{Nonlocal}(k,r) f_{Nonlocal}(-k,s)] \end{aligned} \tag{6}$$

The solutions of Yamaguchi potential (1954) are written as

$$\begin{aligned} \Phi_{Nonlocal}(k,r) = & \frac{\sin(kr)}{k} + \frac{\lambda}{D(k).(\beta^2 + k^2)^2} \\ & \left[e^{-\beta r} + \frac{\beta}{k}\sin(kr) - \cos(kr)\right] \end{aligned} \tag{7}$$

and

$$f_{Nonlocal}(k,r) = e^{ikr} + \frac{\lambda(\beta + ik)e^{-\beta r}}{(\beta^2 + k^2)^2 \, D(k)}$$

(8)

Using the above formalism eq. (1) becomes

$$V_{12}(r) = V(r) - N_1 / D_1 + N_2 / D_2 \tag{9}$$

where,

$$N_1 = \lambda \begin{bmatrix} 2X\beta e^{-\beta r}(1 - e^{-\beta r}) + X_1\big(3e^{-3\beta r}(3\beta r - 1) \\ + e^{-\beta r}(\beta r - 1) + 2e^{-\beta r}(1 - 2\beta r)\big) \end{bmatrix}, \tag{10}$$

$$X_1 = \frac{\lambda}{\left(\beta^2 + k^2 - \lambda / 2\beta\right)} X \ , \tag{11}$$

$$X = \frac{\lambda^2}{(D(k))^2 (\beta^2 + k^2)^4} \ , \tag{12}$$

$$D(k) = 1 - \frac{\lambda}{2\beta(\beta^2 + k^2)} \ , \tag{13}$$

$$D_1 = 1 + \lambda \begin{bmatrix} X(2\beta)^{-1}\left(e^{-2\beta r} + 4e^{-\beta r} + 2\beta r - 3\right) + \\ X_1\left(6\beta^2\right)^{-1}\{e^{-3\beta r}(6\beta r + 2) - e^{-2\beta r} \times \\ (6\beta r + 3) + e^{-\beta r}(6\beta r + 6) - 5\} \end{bmatrix} \tag{14}$$

$$N_2 = \left[\lambda X\left(e^{-2\beta r} - 2e^{-\beta r} + 1\right) + X_1 r\left(2e^{-2\beta r} - 3e^{-3\beta r} - e^{-\beta r}\right)\right]^2 \tag{15}$$

and

$$D_2 = D_1^2 \,. \tag{16}$$

Now we shall calculate the n-p, p-p phase shifts for the equivalent potential $V_{12}(r)$ using phase function method. The phase function method (PFM) (Calogero, 1967) is an efficient tool for calculating the scattering phase shifts for local (Calogero, 1967) and nonlocal interactions (Sett et al., 1988; Bhoi & Laha, 2013; Laha & Bhoi, 2013; Talukdar et al., 1977) in quantum mechanics.

In this case the radial wave function is separated into two parts namely, an amplitude part and an oscillating part with a variable phase $\delta_\ell(k,r)$. For a local potential $\delta_\ell(k,r)$ satisfies a first order non-linear differential equation given by

$$\delta'_\ell(k,r) = -V(r)/k\left[\hat{j}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)\right]^2 . \tag{17}$$

Here $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ stand for the Riccati Bessel functions.

3. RESULTS AND DISCUSSIONS

The nucleon-nucleon scattering phase shifts δ_{np} and δ_{pp} for n-p & p-p systems with the energy-dependent interaction $V_{12}(r)$ for the states $^1S_0$ & $^3S_1$ are computed and compared our results with the standard data of Arndt et al. (1983). To do so we have worked with the parameters of Arnold-MacKellar (1971), van Haeringen (1975) and Laha-Bhoi (2014). We have used $\hbar^2/m_p = 41.47\,MeV\ fm^2$ and $V_0 a = 1/28.8\ fm^{-1}$ for the nucleon-nucleon system. The associated phase shifts are portrayed in Figs. 1-3 along with nucleon-nucleon experimental data (Arndt et al. 1983). Here we have used the parameters of refs. (Arnold & MacKellar 1971), (van Haeringen 1975) and (Laha-Bhoi 2014) which are $\lambda = -5.237\ fm^{-3}$, $\beta = 1.4054\ fm^{-1}$; $\lambda = -2.405\ fm^{-3}$, $\beta = 1.1\ fm^{-1}$ for $^1S_0$ state and $\lambda = -7.533\ fm^{-3}$, $\beta = 1.4054\ fm^{-1}$; $\lambda = -3.901\ fm^{-3}$, $\beta = 1.1\ fm^{-1}$ for $^3S_1$ state respectively.

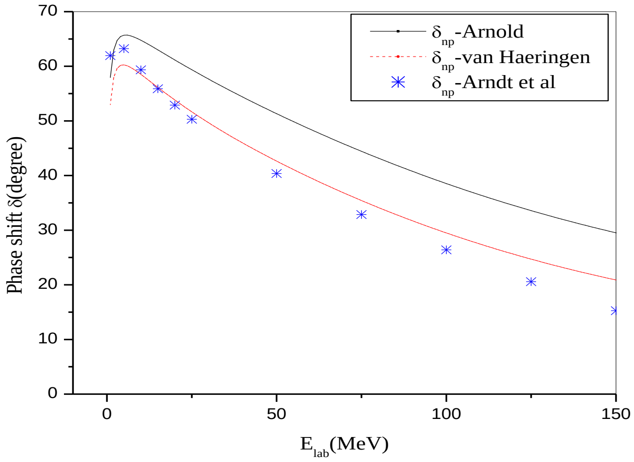

Figure 1. $^1S_0$ n-p Phase shifts as a function of E_{Lab}.

Looking closely in figs. 1 & 2 it is noticed that the $^1S_0$ n-p phase shifts with $V_{12}(r)$ interaction produce better result for the parameters of van Haeringen (1975) than Arnold & MacKellar (1971). For p-p scattering also the parameters of van Haeringen (1975) are in good agreement with the experimental data (Arndt et al. 1983) except at very low energies.

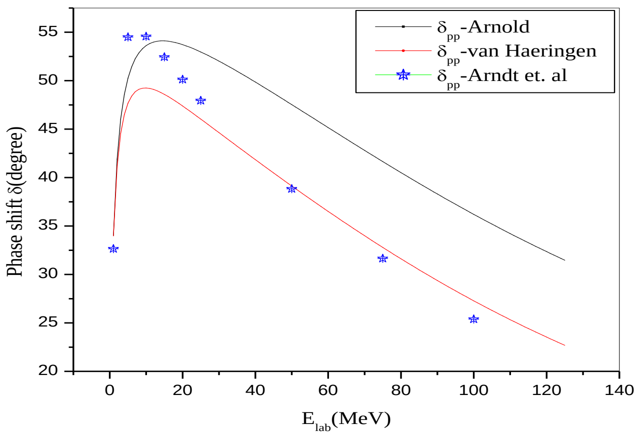

Figure 2. $^{1}S_{0}$ p-p Phase shifts as a function of E_{Lab}

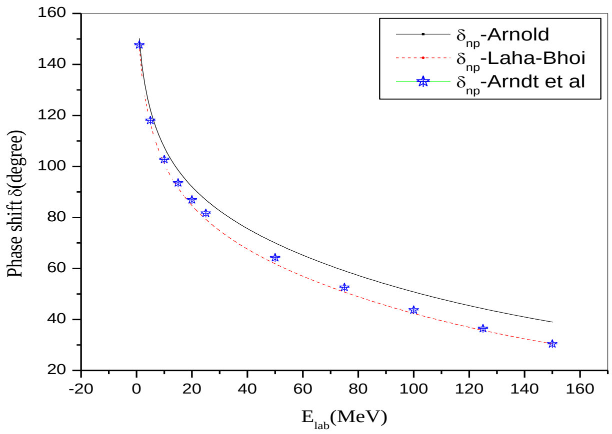

Figure 3. $^{3}S_{1}$ n-p Phase shifts as a function of E_{Lab}.

It may thus be concluded that the parameters of van Haeringen (1975) are superior to Arnold-MacKellar (1971) as far as n-p scattering is concerned. For $^{3}S_{1}$ n-p scattering phases as shown in fig. 3 it is observed that the parameters of Laha-Bhoi (2014) reproduced correct results over the entire range of energies. Therefore, parameters of Laha-Bhoi (2014) are more in agreement with the n-p scattering model instead of Arnold & MacKellar (1971) in $^{3}S_{1}$ state.

4. CONCLUSIONS

In this text first we have localized the separable non-local interaction by the use of the Green's functions with irregular boundary conditions and secondly, used these local wave functions to construct another phase equivalent potential following SUSY approach. The PFM (Calogero, 1967) has been exploited to compute nucleon-nucleon scattering phase shifts for the newly constructed potential.

It is observed that the phase shifts of our energy-momentum dependent equivalent local potential derived via the SUSY formalism are in good agreement with the experimental data (Arndt et al. 1983) particularly, at low and intermediate range of energies.

The phase shifts for the parent nonlocal potential (Yamaguchi 1954) with the parameters in the text match quite well with Arndt et al. (1983) up to maximum 30 MeV and beyond that they discerns. But it is noticed that our energy-momentum dependent equivalent potential, developed via the SUSY formalism, reproduced much better result than the parent nonlocal interaction with the parameters of van Haeringen (1975) and Laha-Bhoi (2014). This is attributed to the fact that the supposition of non-local potential which are originated from a many body theory is unrealistic. As we are habituated to thinking potential locally, equivalent local potential may be a convenient way to explain the properties of non-local one better. Therefore, this attempt has been made towards the development of equivalent local potential to a nonlocal one which provides a greater understanding of nuclear interaction.

Thus, one may conclude that nuclear forces are predominantly non central in character than central one.

Therefore, the combined approach of supersymmetry and the PFM to potential scattering has immense importance to analyze the nucleon-nucleon scattering phase shifts. The present treatment is more acceptable and straightforward than the earlier approaches to the problem (Sukumar 1987; Baye 1987).

The present prescription can also be extended for nucleon-nucleus $(\alpha - n)/(\alpha - p)$ and nucleus-nucleus $(\alpha - \alpha)/(\alpha - H^3)$ elastic scattering without any restriction to the form factors of the nonlocal separable potential. Since our approach to the problem of interest is a simple but different one in the literature for the calculation of physical observables so we think that it must gives some attractive insight for different work to the physicist.

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